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Quantum Mechanics as Relativistic Statistics. II: The Case of Two Interacting Particles

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Abstract

It is shown that non-relativistic quantum mechanics of two particles interacting with external electromagnetic field or between each other can be considered as statistics of two-dimensional surfaces. These surfaces represent the relativistical state of two indeterministic particles in the eight-dimensional space which is a tensor product of space-times for each of the two particles.

In this paper the conception suggested in an earlier work (Rylov, 1971) is extended on the case of two non-relativistical particles interacting with an electromagnetic field or each other.†

According to this conception quantum mechanics is a variety of relativistic statistics. In the present paper it will be shown that the quantum mechanics of two interacting particles can be presented as the statistics of two-dimensional surfaces representing the *r*-state‡ of two particles in eight-dimensional space V_{12} which is the tensor product of space-times V_1 and V_2 for each of particles.

This conception is expounded in detail in the first part of this paper (see p. 65) and in a previous paper (Rylov, 1971). Here I shall briefly formulate the main idea. The classical particles§ are supposed to interact with the

† A review of papers on the interpretation of quantum mechanics from a classical point of view, together with a comprehensive bibliography, can be found in the paper by Kaliski (1970).

‡ In the present paper two different notions of state are used: *n*-state and *r*-state. The *n*-state (non-relativistical state) is given at a certain moment of time. The evolution of the *n*-state is described by motion equations, and the *n*-state of a particle is its coordinates and momentum. The *r*-state (relativistical state) is given over all space-time, and obeys some equations which describe a part of some of the restrictions imposed upon possible *r*-states. The *r*-state of a particle is the equation of its world-line $q^i = q^i(\tau)$. For more detailed information see Rylov (1971).

§ The particles are classical in the sense that the motion of each particle can be described in terms of a world-line in space-time.

medium (ether) in an unpredictable manner. As a result their behaviour is indeterministic and unpredictable. It can be described only statistically. For this purpose the statistical principle (Rylov, 1973) is used. By means of this principle an indeterministic dynamical system is allied to a deterministic dynamical system—the statistical ensemble. Interrelation between the statistical ensemble and the indeterministic system is established on the basis of the two following properties.

1. The ensemble state is a density of states of systems constituting ensemble.
2. The value of every additive quantity (for instance, energy, momentum) attributed to the ensemble as a dynamical system is a mean value of the same quantity for the system constituting an ensemble (provided there is proper normalisation of the ensemble).

It is found that the Lagrangian for the statistical ensemble can be chosen in such a way, that in the non-relativistic approximation the statistical description is equivalent to the quantum mechanical description. For this to occur it is essential that the particles are described by means of a relativistical notion of state (the r-state).

In Sections 1 and 2 of this paper the properties of ensemble of the two indeterministic interacting particles in external electromagnetic field are studied. In Section 3 energy, momentum and angular momentum of ensemble are introduced, and in Section 4 stationary states of ensemble of two indeterministic particles are discussed.

1. *The Statistical Ensemble for two Particles in the Electromagnetic Field*

Let us consider a system consisting of two particles. The r-state of an Ath particle is described by world-line L_A: $q_A{}^i = q_A{}^i(\tau_A)$ $(i = 0, 1, 2, 3;$ $A = 1, 2)$ in the space-time V_A (τ_A is a parameter along a world-line), $q_A{}^i$ $(i = 0, 1, 2, 3)$ are coordinates in V_A. The r-state of a system of two particles is described by a two-dimensional surface $S = L_1 \otimes L_2$ in an eight-dimensional space $V_{12} = V_1 \otimes V_2$. The symbol $\otimes$ stands for tensor product.

Let us introduce coordinates x^a $(a = 1, 2, \ldots, 8)$ in space V_{12}

$$x^a = x^{\binom{i}{A}} = q_A{}^i, \qquad a = 4(A-1) + i + 1 \tag{1.1}$$

Here, and on subsequent occasions, both the tensor indices $a, b, \ldots$ and the double indices $\binom{i}{A}$ will be used. The connection between them is established by means of

$$a \leftrightarrow \binom{i}{A}, \qquad a = 4(A-1) + i + 1$$
$$(a = 1, 2, \ldots, 8;\ i = 0, 1, 2, 3;\ A = 1, 2) \tag{1.2}$$

Latin tensor indices $a, b, \ldots$ take the values $1, 2, \ldots 8$, Latin tensor indices $i, j, \ldots$ take the values 0, 1, 2, 3, and Greek tensor indices take the values 1, 2, 3. As usual, summation is made on like-tensor indices. Summation on capital indices, which numerate particles, is always denoted by the sign of summation.

As was shown earlier (Rylov, 1973) the density of the state surface in the vicinity of point x of space V_{12} is determined by the antisymmetrical tensor $j^{ab}(x)$. We consider the case when $S = L_1 \otimes L_2$

$$j^{\binom{i}{1}\binom{k}{1}} = j^{\binom{i}{2}\binom{k}{2}} = 0 \qquad (i, k = 0, 1, 2, 3) \tag{1.3}$$

According to the statistical principle, the density j^{ab} of state surfaces S is a state of statistical ensemble of a two-particle system. In the case when particles interact only with the external electromagnetic field, the action for statistical ensemble (quantum ensemble) can be written in the form of the sum of two terms

$$S = S_{cl} + S_q \tag{1.4}$$

Where S_{cl} is the action for the ensemble of deterministic systems consisting of two particles interacting with the external electromagnetic field. The expression for S_{cl} can be deduced from the action for two particles in the external electromagnetic field

$$S = \sum_{A=1}^{2} \int \left[-m_A c \sqrt{(dq_A{}^i g_{ik} dq_A{}^k)} + \frac{e_A}{c} A_i(q_A)\, dq_A{}^i \right] \tag{1.5}$$

$$g_{ik} = \begin{Vmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \tag{1.6}$$

Here, m_A and e_A are respectively mass and charge of the Ath particle, c is the speed of light and A_i is the four-potential of the external electromagnetic field.

For this purpose the simple ensemble of deterministic systems described by means of action (1.5) is to be considered.† The deduction may be made in the same way as that which was used for free particles (Rylov, 1973). In the non-relativistic approximation one obtains for S_{cl}:

$$S_{cl} = S_{cl}[j^{ab}, p_a, \xi_\alpha{}^B] = \int \sum_{A=1}^{2} \left\{ m_A \frac{j^{\binom{\alpha}{A}b} \eta_b j^{\binom{\alpha}{A}c} \eta_c}{2 j^{\binom{0}{A}d} \eta_d} \right.$$
$$\left. - p_\alpha \eta_b \left(j^{ab} - \frac{\partial^2 J}{\partial \tau_a \partial \eta_b} \right) + \frac{e_A}{c} j^{\binom{i}{A}b} \eta_b A_{\binom{i}{A}}(q_A) \right\} \delta(\eta - C)\, d^8 x \tag{1.7}$$

where

$$\eta = \eta(t_1, t_2), \qquad t_1 = q_1{}^0, t_2 = q_2{}^0, \qquad q_A = \{q_A{}^0, q_A{}^1, q_A{}^2, q_A{}^3\}$$

† By the phrase simple ensemble is meant an ensemble of state surfaces S, where surfaces S do not cross with each other (see Rylov, 1973).

Here, $\eta = C =$ constant represents, to some extent, the arbitrary seven-dimensional surface in space V_{12}. Integration is made over this surface. Quantities j^{ab}, p_a, $\xi_\alpha{}^B$ ($B = 1,2,\ldots 2s$; $\alpha = 1,2,3$) are variables which are variated,

$$J = \sum_{B=1}^{s} \frac{\partial(\tau, \xi_1^{2B-1}, \xi_2^{2B-1}, \xi_3^{2B-1}, \eta, \xi_1^{2B}, \xi_2^{2B}, \xi_3^{2B})}{\partial(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8)} \tag{1.8}$$

$$s > 1, \qquad \tau \equiv \frac{\partial \tau}{\partial x^a}, \qquad \eta_a \equiv \frac{\partial \eta}{\partial x^a}$$

$\partial^2 J/\partial\tau_a \partial\eta_b$ is a partial derivative of J with respect to τ_a and η_b by fixed $\xi^B_{\alpha,a} \equiv \partial\xi_\alpha{}^B/\partial x^a$. $A_{\binom{i}{A}}(q_A)$ is the four-potential of the external electromagnetic field in space V_A. For the case where electromagnetic field is an external one, then

$$A_{\binom{i}{1}}(q) = A_{\binom{i}{2}}(q) = A_i(q) \tag{1.9}$$

i.e. the form of functions $A_{\binom{i}{1}}$ and $A_{\binom{i}{2}}$ is the same, but they depend on different arguments because they are attributed to different space-times V_A. j^{ab} is the density of state surfaces S, $p_a = p_{\binom{i}{A}}(x)$ represents canonical momentum, i.e. the mean value of canonical momentum p_i of the Ath particle at point x. Formally, p_a is a Lagrangian multiplier, introducing the designation

$$j^{ab}\eta_b = \frac{\partial J}{\partial \tau_a} \tag{1.10}$$

If $s = 1$ in (1.8), it is found that $\xi_\alpha{}^1$, $\xi_\alpha{}^2$ are Lagrangian coordinates, i.e. a set of six quantities $\xi_\alpha{}^1$, $\xi_\alpha{}^2$ ($\alpha = 1, 2, 3$) determines the 'number' of the system in ensemble. Thus the choice of S_{cl} in the form (1.7) is not arbitrary.

It should be taken into account that consideration of the simple ensemble leads, by necessity, to $s = 1$ in (1.8). In addition, η is an arbitrary function of x but not only of t_1 and t_2. In this sense, consideration of a non-simple ensemble and refusal from condition $s = 1$ in (1.8) are some generalisation of result obtained from (1.5). I shall not discuss here necessity of introduction of non-simple ensemble. This is considered in a previous article (Rylov, 1973).

In contrast to (1.7), the introduction of action S_q in (1.4) is a special assumption. S_q takes into account indeterminism of motion of individual system. As result of this indeterminism, 'diffusion' of surfaces S from one region of space into another appears. The intensity of this process is in direct proportion to the gradient of the density j^{ab} of system states. Accordingly, the Langrangian is in proportion to the square of the gradient of density j^{ab}, the proportionality factor being $\hbar^2/(8m)$ ($\hbar$ is Planck's constant). In reality S_q contains the gradient of component $j^{\binom{0}{1}\binom{0}{2}}$ only, because in the non-relativistical case the rest of the components are smaller than $j^{\binom{0}{1}\binom{0}{2}} = j^{\binom{0}{2}\binom{0}{1}}$. I shall choose S_q in the form

$$S_q = S_q[j^{ab}] = -\int \sum_{A=1}^{2} \frac{\hbar^2}{8m_A} \frac{(\partial j^{\binom{0}{A}b}/\partial q_A{}^\alpha)\eta_b(\partial j^{\binom{0}{A}c}/\partial q_A{}^\alpha)\eta_c}{j^{\binom{0}{A}d}\eta} \delta(\eta - C)\, d^8 x \tag{1.11}$$

Thus, (1.11) is a special assumption and its correctness is to be justified by obtained results. I shall call the ensemble whose action contains term S_q the quantum ensemble.

Let us take into account that due to $\eta = \eta(t_1, t_2)$

$$\eta_{\binom{\alpha}{A}} \equiv \frac{\partial \eta}{\partial q_A{}^\alpha} = 0 \qquad (\alpha = 1, 2, 3;\ A = 1, 2) \tag{1.12}$$

and by means of (1.3) write the action in the form

$$S = S_m + S_{m\gamma} + S_q \tag{1.13}$$

$$S_m = S_m[j^{ab}, p_a, \xi_\alpha{}^B] = \int \sum_{A=1}^{2} \left\{ m_A \frac{j^{\binom{\alpha}{A}\binom{0}{3-A}} j^{\binom{\alpha}{A}\binom{0}{3-A}}}{2 j^{\binom{0}{A}\binom{0}{3-A}}} \eta_{\binom{0}{3-A}} - \sum_{B=1}^{2} p_{\binom{i}{A}} \eta_{\binom{0}{B}} \left[(1 - \delta_{AB}) j^{\binom{i}{A}\binom{0}{3-A}} - \frac{\partial^2 J}{\partial \tau_{\binom{i}{A}} \partial \eta_{\binom{0}{B}}} \right] \right\} \delta(\eta - C)\, d^8 x \tag{1.14}$$

$$S_{m\gamma} = S_{m\gamma}[j^{ab}] = \int \sum_{A=1}^{2} \frac{e_A}{c} j^{\binom{i}{A}\binom{0}{3-A}} \eta_{\binom{0}{3-A}} A_i(q_A)\, \delta(\eta - C)\, d^8 x \tag{1.15}$$

$$S_q = S_q[j^{ab}] = -\int \sum_{A=1}^{2} \frac{\hbar^2}{8m_A} \frac{\partial j^{\binom{0}{A}\binom{0}{3-A}}/\partial q_A{}^\alpha\, \partial j^{\binom{0}{A}\binom{0}{3-A}}/\partial q_A{}^\alpha}{j^{\binom{0}{A}\binom{0}{3-A}}} \eta_{\binom{0}{3-A}} \delta(\eta - C)\, d^8 x \tag{1.16}$$

Varying (1.13) with respect to p_a and taking into account the arbitrariness of $\eta(t_1, t_2)$ one obtains

$$j^{\binom{i}{A}\binom{0}{3-A}} = \frac{\partial^2 J}{\partial \tau_{\binom{i}{A}} \partial \eta_{\binom{0}{3-A}}}, \qquad \frac{\partial^2 J}{\partial \tau_{\binom{i}{A}} \partial \eta_{\binom{0}{A}}} = 0 \qquad (A = 1, 2) \tag{1.17}$$

Let us introduce the designations

$$\rho = j^{\binom{0}{1}\binom{0}{2}}, \qquad \rho v_1{}^\alpha = j^{\binom{\alpha}{1}\binom{0}{2}}, \qquad \rho v_2{}^\alpha = j^{\binom{0}{1}\binom{\alpha}{2}} \qquad (\alpha = 1, 2, 3) \tag{1.18}$$

Varying with respect to j^{ab} and taking into account the arbitrariness of $\eta(t_1, t_2)$, one obtains

$$p_{\binom{\alpha}{A}} = m_A v_A{}^\alpha + \frac{e_A}{c} A_\alpha(q_A) \qquad (A = 1, 2;\ \alpha = 1, 2, 3) \tag{1.19}$$

$$p_{\binom{0}{A}} = -m_A \frac{v_A{}^\alpha v_A{}^\alpha}{2} + \frac{\hbar^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q_A{}^\alpha \partial q_A{}^\alpha} + \frac{e_A}{c} A_0(q_A) \qquad (A = 1, 2) \tag{1.20}$$

The relations (1.19) and (1.20) can be obtained from the corresponding relation for uncharged particles by means of the substitution

$$p_{\binom{i}{A}} \to p_{\binom{i}{A}} + \frac{e_A}{c} A_i(q_A) \tag{1.21}$$

Let us numerate all $\xi_\alpha{}^B$ by means of one index $n = 1, 2, \ldots, 6s + 2$ ($\xi_1 = \tau$, $\xi_2 = \eta$). Varying with respect to ξ_n ($n = 3, 4, \ldots, 6s + 2$) leads to the equation

$$\frac{\delta S}{\delta \xi_n} = \delta(\eta - C) \sum_{A,B=1}^{2} \frac{\partial}{\partial x^a} \left(p_{\binom{i}{A}} \eta_{\binom{0}{B}} \frac{\partial^3 J}{\partial \tau_{\binom{i}{A}} \partial \eta_{\binom{0}{B}} \partial \xi_{n,a}} \right) = 0$$

$$(n = 3, 4, \ldots, 6s + 2) \tag{1.22}$$

Equation (1.22) is the identity for $n = 1, 2$, and therefore it is right for $n = 1, 2, \ldots, 6s + 2$.

Using identities

$$\frac{\partial}{\partial x^a} \frac{\partial^3 J}{\partial \tau_c \partial \eta_b \partial \xi_{n,a}} = 0 \tag{1.23}$$

$$\sum_{n=1}^{6s+2} \frac{\partial^3 J}{\partial \tau_c \partial \eta_b \partial \xi_{n,a}} \xi_{n,d} = \delta_d{}^a \frac{\partial^2 J}{\partial \tau_c \partial \eta_b} + \delta_d{}^b \frac{\partial^2 J}{\partial \tau_a \partial \eta_c} + \delta_d{}^c \frac{\partial^2 J}{\partial \tau_b \partial \eta_c} \tag{1.24}$$

one obtains from (1.22) after calculation

$$\frac{\partial p_{\binom{\alpha}{A}}}{\partial t_B} = \frac{\partial p_{\binom{0}{B}}}{\partial q_A{}^\alpha} + v_B{}^\beta \left(\frac{\partial p_{\binom{\beta}{B}}}{\partial q_A{}^\alpha} - \frac{\partial p_{\binom{\alpha}{A}}}{\partial q_B{}^\beta} \right), \qquad A, B = 1, 2. \tag{1.25}$$

$p_{\binom{0}{A}}$ is a function of $p_{\binom{\alpha}{A}}$, whose form can be determined from (1.19) and (1.20). Equation (1.25) has the potential solution

$$\partial_a p_c = \partial_c p_a, \qquad p_a = \frac{\partial \phi}{\partial x^a} \qquad (a, c = 1, 2, \ldots, 8) \tag{1.26}$$

where ϕ is an arbitrary function of x.

Let us consider the case of potential solution. Substituting (1.26) into (1.19), (1.20) and eliminating $v_A{}^\alpha$, one obtains

$$\frac{\partial \phi}{\partial t_A} + \frac{1}{2m_A} \left[\frac{\partial \phi}{\partial q_A{}^\alpha} - \frac{e_A}{c} A_\alpha(q_A) \right] \left[\frac{\partial \phi}{\partial q_A{}^\alpha} - \frac{e_A}{c} A_\alpha(q_A) \right]$$

$$- \frac{\hbar^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q_A{}^\alpha \partial q_A{}^\alpha} - \frac{e_A}{c} A_0(q_A) = 0 \qquad (A = 1, 2) \tag{1.27}$$

By means of (1.17), (1.18), (1.19) and (1.26) the identity

$$\frac{\partial}{\partial x^b}\frac{\partial^2 J}{\partial \tau_b \, \partial \eta_{\binom{0}{A}}} = 0 \tag{1.28}$$

can be written in the form

$$\frac{\partial \rho}{\partial t_A} + \frac{\partial}{\partial q_A{}^\alpha}\left(\frac{\rho}{m_A}\frac{\partial \phi}{\partial q_A{}^\alpha} - \frac{e_A}{m_A c}\rho A_\alpha(q_A)\right) = 0 \qquad (A = 1, 2) \tag{1.29}$$

The quantities $j^{\binom{\alpha}{1}\binom{\beta}{2}}$ $(\alpha, \beta = 1,2,3)$ remain indefinite. They can be defined by means of

$$j^{\binom{\alpha}{1}\binom{\beta}{2}} = \rho v_1{}^\alpha v_2{}^\beta - \frac{\hbar^2}{4m_1 m_2}\left(\frac{\partial^2 \rho}{\partial q_1{}^\alpha \, \partial q_2{}^\beta} - \frac{1}{\rho}\frac{\partial \rho}{\partial q_1{}^\alpha}\frac{\partial \rho}{\partial q_2{}^\beta}\right) \qquad (A = 1, 2) \tag{1.30}$$

Then due to (1.3) the 'conservation laws' are fulfilled

$$\frac{\partial}{\partial x^a} j^{ab} = 0 \qquad (b = 1, 2, \ldots, 8) \tag{1.31}$$

Let us multiply (1.30) by $-\sqrt{\rho}\exp(i\phi/\hbar)$ and (1.32) by $i\hbar\exp(i\phi/\hbar)/(2\sqrt{\rho})$ and add them. One obtains

$$\left(i\hbar\frac{\partial}{\partial t_A} + \frac{e_A}{c}A_0(q_A)\right)\psi - \frac{1}{2m_A}\left(i\hbar\frac{\partial}{\partial q_A{}^\alpha} + \frac{e_A}{c}A_\alpha(q_A)\right) \times \left(i\hbar\frac{\partial}{\partial q_A{}^\alpha} + \frac{e_A}{c}A_\alpha(q_A)\right)\psi = 0 \qquad (A = 1, 2) \tag{1.32}$$

where

$$\psi = \sqrt{\rho}\exp(i\phi\hbar) \tag{1.33}$$

It is easy to see, that two equations (1.32) are always compatible. Equations (1.27) and (1.29) are equivalent to (1.32) and also always compatible. Equations (1.32) describe the evolution of ψ with respect to two times.

Let us take now the consistent non-relativistic point of view. This means that the ensemble is described only at equal times $t_1 = t_2$, i.e. in the seven-dimensional plane P_7 of space V_{12}. Making the transformation

$$t = \frac{t_1 + t_2}{2}, \qquad T = \frac{t_1 - t_2}{2} \tag{1.34}$$

one obtains instead of (1.35) two equations

$$i\hbar\frac{\partial \psi}{\partial t} + \sum_{A=1}^{2}\left\{\frac{e_A}{c}A_0(q_A) + \frac{\hbar^2}{2m_A}\left(\frac{\partial}{\partial q_A{}^\alpha} - \frac{ie_A}{\hbar c}A_\alpha(q_A)\right) \times \left(\frac{\partial}{\partial q_A{}^\alpha} - \frac{ie_A}{\hbar c}A_\alpha(q_A)\right)\right\}\psi = 0 \tag{1.35}$$

$$i\hbar\frac{\partial\psi}{\partial T}+\sum_{A=1}^{2}(-1)^{A-1}\left\{\frac{e_A}{c}A_0(q_A)+\frac{\hbar^2}{2m_A}\left(\frac{\partial}{\partial q_A{}^\alpha}-\frac{ie_A}{\hbar c}A_\alpha(q_A)\right)\right.$$
$$\left.\times\left(\frac{\partial}{\partial q_A{}^\alpha}-\frac{ie_A}{\hbar c}A_\alpha(q_A)\right)\right\}\psi=0 \quad (1.36)$$

Equation (1.35) is Schrödinger's equation for two particles in an external electromagnetic field. It contains T as a parameter. If function ψ is given at $t=0$ and $T=0$, it can be determined at any t and $T=0$ by means of only equation (1.35).

In the plane P_7 the system state is represented by line but not by two-dimensional surface. For this reason state density is represented by the vector j^i $(i=0, 1, \ldots, 6)$. In the coordinate system $y^0=t$, $y^i=q_A{}^\alpha$ $[i=3(A-1)+\alpha;\ \alpha=1,2,3;\ A=1,2]$ j^i has the form

$$j^i=\sqrt{(2)}\{\rho,\rho v_1{}^\alpha,\rho v_2{}^\alpha\} \quad (1.37)$$

This can be shown by employing the method which was used in a previous article (Rylov, 1973). It follows from the conservation laws (1.29) that

$$\sum_{i=0}^{6}\frac{\partial}{\partial y^i}j^i=0 \quad (1.38)$$

If j^i has proper normalisation then it follows from (1.37) that j^0 is a probability density to detect the first particle at the point $\mathbf{q}_1$ and the second one at the point $\mathbf{q}_2$. The rest of components of j^i describe spatial components of probability flux. They are expressed through wave function according to quantum mechanics formulae.

2. *Ensemble of Interacting Particles*

Let us consider now the case of two charged interacting particles in the absence of external field. This means that in the action (1.5) a four-potential acting on the first particle is conditioned by the second particle, and vice versa. Strictly speaking, degrees of freedom connected with electromagnetic field are to be taken into account. But only the non-relativistic case will be considered, and therefore radiation is neglected.

For the determination of the Lagrangian of system of two interacting particles the four-potential A_i is supposed to be conditioned by the charges of particles. Also the term omitted in (1.5) describing the free electromagnetic field is to be taken into account. Due to Maxwell equations, it can be written in the form

$$-\frac{1}{16\pi}\int(\partial_i A_k(q)-\partial_k A_i(q))(\partial^i A^k(q)-\partial^k A^i(q))\,d^4q$$
$$=-\tfrac{1}{2}\sum_{A=1}^{2}\int e_A A_i(q_A)\frac{dq_A{}^i}{d\tau}d\tau \quad (2.1)$$

Taking into account this term and Maxwell equations, the action (1.5) takes in the non-relativistical approximation ($c \to \infty$) the following form

$$\int \left(\sum_{A=1}^{2} \frac{\mu_A \dot{z}_A^{\alpha} \dot{z}_A^{\alpha}}{2\dot{z}_A^0} - \frac{e_1 e_2}{R_{12}} \dot{z}_2^0 \right) d\tau, \qquad \dot{z}_A^i \equiv \frac{dz_A^i}{d\tau} \tag{2.2}$$

where

$$z_1^{\alpha} = \frac{m_1 q_1^{\alpha} + m_2 q_2^{\alpha}}{m_1 + m_2}, \qquad z_2^{\alpha} = q_1^{\alpha} - q_2^{\alpha} \qquad (\alpha = 1,2,3)$$

$$z_1^0 = q_1^0, \; z_2^0 = q_2^0, \qquad \mu_1 = m_1 + m_2, \qquad \mu_2 = \frac{m_1 m_2}{m_1 + m_2}$$

$$R_{12} = \sqrt{(z_2^{\alpha} z_2^{\alpha})} \tag{2.3}$$

Let us consider a simple ensemble consisting of systems described by action (2.2). Let $\xi = \{\xi_1^1, \xi_2^1, \xi_3^1, \xi_1^2, \xi_2^2, \xi_3^2\}$ numerate the systems of ensemble, $z_A^i = q_A^i(\tau, \xi, \eta)$, where η is a parameter taking the same value C for all systems in the ensemble. Then

$$S = S[z_A^i] = \int \left(\sum_{A=1}^{2} \frac{\mu_A}{2} \frac{\partial z_A^{\alpha}/\partial\tau \, \partial z_A^{\alpha}/\partial\tau}{\partial z_A^0/\partial\tau} - \frac{e_1 e_2}{R_{12}} \frac{\partial z_2^0}{\partial \tau} \right)$$

$$\times \, \delta(\eta - C) \, d\tau \, d\eta \, d^6 \xi \qquad \left(d^6 \xi = \prod_{\alpha=1}^{3} \prod_{A=1}^{2} d\xi_{\alpha}^A \right) \tag{2.4}$$

Reverse relations $z_A^i = z_A^i(\tau, \xi, \eta)$ and consider (2.4) as a functional of $\tau, \xi, \eta = \tau, \xi, \eta(z_A^i)$. Transforming (2.4) one obtains

$$S = S[\xi_{\alpha}^B] = \int \left(\sum_{A=1}^{2} \frac{\mu_A}{2} \frac{j^{\binom{\alpha}{A}a} j^{\binom{\alpha}{A}b} \eta_a \eta_b}{j^{\binom{0}{A}c} \eta_c} - \frac{e_1 e_2}{R_{12}} j^{\binom{0}{2}b} \eta_b \right) \delta(\eta - C) \, d^8 x \tag{2.5}$$

where

$$j^{ab} = \frac{\partial^2 J}{\partial \tau_a \partial \eta_b}, \qquad J = \frac{\partial(\tau, \xi_1^1, \xi_2^1, \xi_3^1, \eta, \xi_1^2, \xi_2^2, \xi_3^2)}{\partial(x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^8)} \tag{2.6}$$

and $x^a = z(_A^i)$ with correspondence between a and $(_A^i)$ given by (1.2). Equation (2.5) is an action for the ensemble of particles which interact in accordance with Coulomb's law. Produce some generalisation in the sense of transition from (2.6) to (1.8) and compare (2.5) with (1.7).

Then taking into account (1.3) and (1.2) one concludes that the interaction of particles is described by the term

$$S_{12} = \int \frac{e_1 e_2}{R_{12}} j^{\binom{0}{2}\binom{0}{1}} \eta_{\binom{0}{1}} \, \delta(\eta - C) \, d^8 x \tag{2.7}$$

Thus the quantum ensemble of two non-relativistical particles interacting in accord with Coulomb's law is described by the action

$$S = S[j^{ab}, p_a, \xi_\alpha{}^B] = S_m + S_{12} + S_q, \tag{2.8}$$

where S_m, S_{12}, S_q are given by expressions (1.14), (1.16) and (2.7) respectively.

Varying with respect to p_a and $\xi_\alpha{}^B$ leads to the former equations (1.17) and (1.22). Varying with respect to $j^{\binom{i}{A}\binom{0}{3-A}}$ leads to

$$p_{\binom{\alpha}{A}} = \mu_A v_A{}^\alpha \qquad (A = 1, 2) \tag{2.9}$$

$$p_{\binom{0}{A}} = -\mu_A \frac{v_A{}^\alpha v_A{}^\alpha}{2} + \frac{\hbar^2}{2\mu_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial z_A{}^\alpha \partial z_A{}^\alpha} - \frac{e_1 e_2}{R_{12}} \delta_{A2} \qquad (A = 1, 2) \tag{2.10}$$

Repeating all calculations which have brought us from (1.17), (1.19)–(1.22) to (1.32), one gets instead of (1.32)

$$\left(i\hbar \frac{\partial}{\partial t_A} - \frac{e_1 e_2}{R_{12}} \right) \delta_{A2} \psi + \frac{\hbar^2}{2\mu_A} \frac{\partial^2 \psi}{\partial z_A{}^\alpha \partial z_A{}^\alpha} = 0 \qquad (A = 1, 2) \tag{2.11}$$

Two equations (2.11) are compatible because of

$$\frac{\partial}{\partial z_1{}^\alpha} \frac{1}{R_{12}} = 0 \qquad \alpha = 1, 2, 3 \tag{2.12}$$

Instead of (1.35) and (1.36) one obtains

$$i\hbar \frac{\partial \psi}{\partial t} - \frac{e_1 e_2}{R_{12}} + \sum_{A=1}^{2} \frac{\hbar^2}{2\mu_A} \frac{\partial^2 \psi}{\partial z_A{}^\alpha \partial z_A{}^\alpha} = 0 \tag{2.13}$$

$$i\hbar \frac{\partial \psi}{\partial T} + \frac{e_1 e_2}{R_{12}} + \sum_{A=1}^{2} (-1)^{A-1} \frac{\hbar^2}{2\mu_A} \frac{\partial^2 \psi}{\partial z_A{}^\alpha \partial z_A{}^\alpha} = 0 \tag{2.14}$$

Returning to variables $q_A{}^i$ we get instead of (2.13)

$$i\hbar \frac{\partial \psi}{\partial t} - \frac{e_1 e_2}{R_{12}} + \sum_{A=1}^{2} \frac{\hbar^2}{2m_A} \frac{\partial^2 \psi}{\partial q_A{}^\alpha \partial q_A{}^\alpha} = 0 \tag{2.15}$$

Equation (2.15) is Schrödinger's equation for two nonrelativistic particles interacting in accordance with Coulomb's law.

3. *Energy, Momentum and Angular Momentum of the Quantum Ensemble*

Quantum ensemble as a dynamical system can be attributed to energy, momentum and angular momentum. These quantities can be obtained from Lagrangian by the canonical way.

Let the action (1.13) be determined as an integral over a certain region Ω of space V_{12}

$$S = \int_{\Omega} L d^8 x \tag{3.1}$$

Making an infinitesimal transformation of coordinates

$$x^a \to x^a + \delta x^a \tag{3.2}$$

one sees in the case when $\delta x^a = \text{constant}$ that the transformation (3.2) induces a variation of action (3.1)

$$\delta S = -\int_{\Sigma} T_b^{ac}\, \delta x^b \eta_c \delta(\eta - C)\, ds_a \tag{3.3}$$

where Σ is a seven-dimensional surface, bounding volume Ω. ds_a is an element of this surface.

Here

$$T_b^{ac} \eta_c \delta(\eta - C) = \sum_{\gamma} \frac{\partial L}{\partial u_{\gamma,a}} u_{\gamma,b} - \delta_b{}^a L \tag{3.4}$$

where $u_\gamma = \{j^{ab}, p_a, \xi_\alpha{}^B, \eta\}$, $u_{\gamma,a} \equiv \partial u_\gamma / \partial x^a$ and summation is made over all indices γ numerating variables (including η). The fact that the left-hand side of (3.4) can be written in the form $T_b^{ac}\eta_c$ is conditioned by a specific form of Lagrangian which is defined by (1.13)–(1.16).

In the case where the volume Ω is bounded by two planes $t_1 = T_1 =$ constant, $t_1 = T_2 = \text{constant}$ $(T_2 > T_1)$, then choosing $\eta(t_1, t_2) = t_2 - t_1$, $C = 0$ one gets for (3.3)

$$\delta S = -\int_{t_1=t_2=T_2} T_b^{\binom{0}{1}\binom{0}{2}} \delta x^b\, d\mathbf{q}_1\, d\mathbf{q}_2 + \int_{t_1=t_2=T_1} T_b^{\binom{0}{1}\binom{0}{2}} \delta x^b\, d\mathbf{q}_1\, d\mathbf{q}_2,$$

$$d\mathbf{q}_A = dq_{4A}{}^1\, dq_A{}^2\, dq_A{}^3 \qquad (A = 1, 2) \tag{3.5}$$

Vector

$$P_b = \int T_b^{\binom{0}{1}\binom{0}{2}} d\mathbf{q}_1\, d\mathbf{q}_2 = \int T_b^{15}\, d\mathbf{q}_1\, d\mathbf{q}_2 \qquad (b = 1, 2, \ldots, 8) \tag{3.6}$$

represents the energy-momentum vector and is conserved for ensemble of free particles. T_b^{ac} represents the energy-momentum tensor. The fact that this is a tensor of the third rank, not, as is usually the case, the second one, is connected with the presence of two-time description.

Calculation for the energy-momentum density of system described by (1.13) results in

$$T^{15}_{\binom{i}{A}} = -p_{\binom{i}{A}} \rho \tag{3.7}$$

where $p_{\binom{\alpha}{A}}$ is defined by expression (1.19) and

$$p_{\binom{0}{A}} = -m_A \frac{v_A{}^\alpha v_A{}^\alpha}{2} + \frac{\hbar^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q_A{}^\alpha \partial q_A{}^\alpha} + \frac{e_A}{c} A_0(q_A) \tag{3.8}$$

Raising the lower index in (3.7) by means of a five-dimensional metric tensor (see Appendix), results in a gauge-invariant form

$$T^{15,\binom{\alpha}{A}} = m_A v_A{}^\alpha \rho \tag{3.9}$$

$$c^2 T^{15,\binom{0}{A}} = \left(\frac{m_A v_A{}^\alpha v_A{}^\alpha}{2} - \frac{\hbar^2}{2m_A}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial q_A{}^\alpha \partial q_A{}^\alpha}\right)\rho \tag{3.10}$$

Analogously, considering transformation (3.2) which describes infinitesimal rotation in the plane t_A = constant of space V_A, one can introduce the angular momentum

$$M_A^{\alpha\beta} = \int M^{15,\binom{\alpha}{A}\binom{\beta}{A}}\, d\mathbf{q}_1\, d\mathbf{q}_2 \tag{3.11}$$

where

$$M^{15,\binom{\alpha}{A}\binom{\beta}{B}} = \delta_{AB}\{q_B{}^\beta T^{15,\binom{\alpha}{A}} - q_A{}^\alpha T^{15,\binom{\beta}{B}}\} = \delta_{AB} m_A \{q_A{}^\beta v_A{}^\alpha - q_A{}^\alpha v_A{}^\beta\}\rho \tag{3.12}$$

Let us introduce operators

$$\hat{p}_{\binom{\alpha}{A}} = -i\hbar\frac{\partial}{\partial q_A{}^\alpha}, \qquad \hat{p}^{\binom{\alpha}{A}} = i\hbar\frac{\partial}{\partial q_A{}^\alpha} + \frac{e_A}{c}A_\alpha(q_A) \tag{3.13}$$

Let us suppose that condition (1.26) is fulfilled, then

$$P^{\binom{\alpha}{A}} = -\int \psi^* \hat{p}^{\binom{\alpha}{A}} \psi\, d\mathbf{q}_1\, d\mathbf{q}_2 \qquad (A = 1, 2; \qquad \alpha = 1, 2, 3) \tag{3.14}$$

$$c^2 P^{\binom{0}{A}} = \int \psi^* \frac{\hat{p}^{\binom{\alpha}{A}}\hat{p}^{\binom{\alpha}{A}}}{2m_A} \psi\, d\mathbf{q}_1\, d\mathbf{q}_2 \qquad (A = 1, 2) \tag{3.15}$$

$$M_A^{\alpha\beta} = \int \psi^*(q_A{}^\alpha \hat{p}^{\binom{\beta}{A}} - q_A{}^\beta \hat{p}^{\binom{\alpha}{A}})\psi\, d\mathbf{q}_1\, d\mathbf{q}_2 \qquad (A = 1, 2) \tag{3.16}$$

where ψ^* is complex conjugate to ψ, and ψ is defined by (1.33).

In the case when particles are uncharged ($e_1 = e_2 = 0$) all quantities $P^{\binom{\alpha}{A}}$, $c^2 P^{\binom{0}{A}}$, $M_A^{\alpha\beta}$ are conserved. So far as $P^{\binom{\alpha}{A}}$, $c^2 P^{\binom{0}{A}}$, $M_A^{\alpha\beta}$ are additive and are connected correspondently with spatial translation, time translation and spatial rotation, according to statistical principle they can be treated as mean momentum of the Ath particle, mean energy of the Ath particle and mean angular momentum of the Ath particle respectively.

Formulae (3.14)–(3.16) coincide with the calculation rule of the mean value of these quantities in quantum mechanics, if ψ is normed in accordance with

$$\int \psi^* \psi\, d\mathbf{q}_1\, d\mathbf{q}_2 = 1 \tag{3.17}$$

If this condition is fulfilled then, due to (1.18), $\rho = \psi^*\psi$ can be treated as a probability density to detect the first particle at the point $\mathbf{q}_1$ and the second one at $\mathbf{q}_2$. For this reason the mean value of arbitrary function $F(\mathbf{q}_1, \mathbf{q}_2)$ is defined by

$$\langle F \rangle = \int \psi^* F(\mathbf{q}_1, \mathbf{q}_2) \psi \, d\mathbf{q}_1 \, d\mathbf{q}_2 \tag{3.18}$$

where angular brackets denote mean value.

4. *Stationary States of Quantum Ensemble*

Let us consider the quantum ensemble of two uninteracting particles in a given external electromagnetic field. Let us suppose that the electromagnetic field is stationary. Then the four-potential A_i can be chosen stationary, i.e.

$$\frac{\partial A_i(q_A)}{\partial t_A} = 0 \qquad [A_i(q_A) = A_i(\mathbf{q}_A)] \tag{4.1}$$

In general, the ensemble state depends on two times t_1 and t_2, or in terms of (1.34) on t and T. Let us consider the ensemble at the same times ($t_1 = t_2$), or $T = 0$. The ensemble state is termed a stationary one if it does not depend on t at $T = 0$, i.e.

$$\frac{\partial j^{ab}}{\partial t} = 0, \qquad \frac{\partial p_a}{\partial t} = 0, \qquad \text{at } T = 0 \qquad (a, b = 1, 2, \ldots, 8) \tag{4.2}$$

In reality the conditions (4.2) are not independent, and due to (1.18), (1.19), (3.8) and (4.1) the second condition (4.2) follows from the first one.

Let us obtain the equation, which is obeyed by the stationary state, supposing that (1.26) is fulfilled. It follows from (1.26) and (4.2) that

$$\phi = \phi_0(\mathbf{q}_1, \mathbf{q}_2) + \phi_1(t) \tag{4.3}$$

One writes the equations (1.27) in the form

$$\begin{aligned}\frac{\partial \phi}{\partial t} = \sum_{A=1}^{2} \bigg\{ &-\frac{1}{2m_A}\left(\frac{\partial \phi}{\partial q_A{}^\alpha} - \frac{e_A}{c} A_\alpha(q_A)\right)\left(\frac{\partial \phi}{\partial q_A{}^\alpha} - \frac{e_A}{c} A_\alpha(q_A)\right) \\ &+ \frac{\hbar^2}{2m_A} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial q_A{}^\alpha \partial q_A{}^\alpha} + \frac{e_A}{c} A_0(q_A) \bigg\}\end{aligned} \tag{4.4}$$

The right-hand side of (4.4) does not depend on t; therefore, $\partial \phi / \partial t$ does not depend on t either. Equation (4.3) takes the form

$$\phi = \phi_0(\mathbf{q}_1, \mathbf{q}_2) - H' t \tag{4.5}$$

where H' is a real constant.

Adding equations (1.29), and taking into account that ρ is independent on t, one obtains

$$\sum_{A=1}^{2} \frac{\partial}{\partial q_A{}^\alpha} \left\{ \frac{\rho}{m_A} \frac{\partial \phi}{\partial q_A{}^\alpha} - \frac{e_A}{m_A c} \rho A_\alpha(q_A) \right\} = 0. \tag{4.6}$$

Combining (4.4) and (4.6) one obtains for ψ

$$\hat{H} \psi = H' \psi \tag{4.7}$$

where $\hat{H}$ is a Hamiltonian operator

$$\hat{H} = \sum_{A=1}^{2} \left\{ \frac{1}{2m_A} \hat{p}^{(\alpha)}_{(A)} \hat{p}^{(\alpha)}_{(A)} - \frac{e_A}{c} A_0(q_A) \right\} \tag{4.8}$$

Thus determination of stationary states of quantum ensemble is reduced to determination of eigenvalues and eigenfunctions of Hamiltonian $\hat{H}$. It is easy to see that the reverse statement is valid as well.

The traditional statistical interpretation of quantum mechanics (Neuman, 1938, Chapter 3, Section 1) can be deduced from the following two statements.

1. If the quantity R corresponds to operator $\hat{R}$ then the quantity $f(R)$ corresponds to operator $f(\hat{R})$
2. The mean value of any quantity R in the state ψ is determined by the relation

$$\langle R \rangle = \int \psi^* \hat{R} \psi \, dN \tag{4.9}$$

The integral in (4.9) denotes integration over all arguments, on which the wave function depends.

The correctness of (4.9) was deduced from relativistic statistics† for additive quantities and an arbitrary function of spatial coordinates only. The relation (4.9) for arbitrary quantity R cannot be deduced from relativistic statistics. And what is more, (4.9) is incompatible with relativistic statistics, because it follows from (4.9) that a particle could not have definite coordinate and definite momentum simultaneously (Moyal, 1949).

In this connection the question arises as to what extent is (4.9) necessary for an explanation of experiment data, and whether it is possible to explain experiment data using relativisitc statistics only. I cannot answer this question comprehensively. I shall make some remarks only.

It follows from (4.9) that measurement of the value of quantity R can give only a value which coincides with one of the eigenvalues of operator $\hat{R}$, corresponding to quantity R. But in reality only those quantities can be measured whose operators commutate with the Hamiltonian of system, and the system state is made stationary as a result of a measurement process. This has been shown by Neumann (1938, Chapter 5, Section 1).

In fact, measurement of any quantity R, concerning the physical system S with wave function ψ, is some influence on system S. As a result of this influence the Hamiltonian H of the system is changed in such a way that it begins to commutate with the operator $\hat{R}$ of quantity R, and the state ψ becomes a stationary state, i.e. an eigenstate of operator $\hat{H}$. This arises from the fact that no measurement is made instantly and the state ψ must be such that it would not be changed for a time of measurement, i.e. the state ψ must be a stationary one. If operator $\hat{R}$ commutates with a Hamiltonian then its eigenvalues R' can serve for a numeration of the eigenvalues of the Hamiltonian.

† I call the conception, proposed in an earlier paper (Rylov, 1971), and developed in the present paper, relativistic statistics.

On the other hand, it follows from relativistic statistics that stationary states are eigenstates of the Hamiltonian. This was shown for the case of two (except measurement of coordinate) particles in an electromagnetic field, and apparently it is correct in other cases. Value R' of any measured quantity R can be considered as a 'label' of a stationary state, and this 'label' can be determined, identifying a stationary state of quantum ensemble.

As an example let us consider the measurement of electron orbital angular momentum in Stern's and Gerlach's experiment. Let there be atoms with non-zero orbital angular momentum of electrons and with vanishing spin (for example an atom of magnesium). In passing a beam of such atoms across an inhomogeneous magnetic field, one can observe splitting of the initial beam into several discrete beams in accordance with discrete eigenvalues of the operator $\hat{M}_H$ of the angular momentum projection on the magnetic field direction. According to the traditional explanation, in this experiment the value of the orbital angular momentum projection M_H is measured. All measured values of the M_H are integer multiplied by $\hbar$. Discreteness of values of M_H is considered as a consequence of discreteness of eigenvalues of operator $\hat{M}_H$.

From the point of view of relativistic statistics the discreteness of eigenvalues of operator $\hat{M}_H$ is in itself of no importance. It is important only that discrete eigenvalues of operator $\hat{M}_H$ numerates discrete stationary states. Atoms in different stationary states get different momenta in an inhomogeneous magnetic field and are separated in space.

From the point of view of relativistic statistics, discreteness of measured values is an attribute of stationary states. From a traditional standpoint, discreteness is an attribute of angular momentum. If the last standpoint is true, then there are to be such experiments, where discrete eigenvalues of angular momentum are measured, but discrete stationary states do not appear. The existence of such experiments is doubtful.

Appendix

A Gauge-Invariant Form of Energy-momentum Tensor for a Particle in an Electromagnetic Field

The particle motion in an electromagnetic field is described by the action

$$S = S_m + S_\gamma = \int L\sqrt{-g}\, d^4 x \tag{A.1}$$

$$S_m = S_m[q^i(\tau), A_k(x)] = \int \{-mc\sqrt{(\dot{q}^i g_{ik}\dot{q}^k)} + \frac{e_A}{c} A_i(q)\dot{q}^i\}\, d\tau, \tag{A.2}$$

$$S_\gamma = S_\gamma[A_k(x)] = -\frac{1}{16\pi}\int F_{ik}F^{ik}\sqrt{-g}\, d^4 x$$

$$F_{ik} = F_{ik}(x) = \partial_i A_k(x) - \partial_k A_i(x) \qquad \left(\dot{q}^i \equiv \frac{dq^i}{d\tau}\right) \tag{A.3}$$

where x^i are certain curvilinear coordinates in space-time, g_{ik} is a metric tensor, and

$$g = \det \|g_{ik}\| \tag{A.4}$$

The energy-momentum tensor can be calculated by two different ways. The first way, variation with respect to g_{ik}, results in

$$T^{ik}(x) = \frac{\delta S}{\delta g_{ik}(x)} = -\frac{2}{\sqrt{-g}}\frac{\partial}{\partial g_{ik}(x)}[\sqrt{(-g)}L]. \tag{A.5}$$

The second way, the canonical one, results in

$$\Theta_k{}^i(x) = \sum_\gamma \frac{\partial L}{\partial u_{\gamma,i}} u_{\gamma,k} - \delta_k{}^i L \tag{A.6}$$

where u_γ are variables, which are to be variable in the action for obtaining motion equations.

The first way results in

$$T_m^{ik}(x) = \frac{mc\dot{q}^i(\tau_0)\dot{q}^k(\tau_0)}{\sqrt{[\dot{q}^l(\tau_0)g_{ls}(x)\dot{q}^s(\tau_0)]}}\frac{\delta(q(\tau_0)-x)}{|\dot{q}^0(\tau_0)|} \tag{A.7}$$

where τ_0 is a root of equation

$$q^0(\tau_0) - x^0 = 0 \tag{A.8}$$

$$T_{\gamma}{}^{ik}_{\cdot\cdot}(x) = -\frac{1}{4\pi}\{F^{il}F^k{}_{.l} - \tfrac{1}{4}g^{ik}F_{jl}F^{jl}\} \tag{A.9}$$

The canonical way results in†

$$\Theta_m{}^{\cdot i}{}_{\cdot k}(x) = \left\{\frac{mc\dot{q}^i g_{kl}\dot{q}^l}{\sqrt{(\dot{q}^j g_{js}\dot{q}^s)}} - \frac{e}{c}A_k(x)\dot{q}^i\right\}\frac{\delta(q-x)}{|\dot{q}^0|} \tag{A.10}$$

argument τ_0 is omitted here.

$$\Theta_\gamma{}^{\cdot i}{}_{\cdot k}(x) = g_{kl}T_\gamma^{il} - \frac{1}{4\pi}\partial_l(A_k F^{il}) + \frac{1}{4\pi}A_k\partial_l F^{il} \tag{A.11}$$

It follows from Maxwell's equations

$$\partial_l F^{il} = \frac{4\pi e}{c}\dot{q}^i\frac{\delta(q-x)}{|\dot{q}^0|} \tag{A.12}$$

that

$$g_{lk}(T_m^{ik} + T_\gamma^{ik}) = \Theta_m{}^{\cdot i}{}_{\cdot l} + \Theta_\gamma{}^{\cdot i}{}_{\cdot l} + \frac{1}{4\pi}\partial_k(A_l F^{ik}) \tag{A.13}$$

Thus different ways of definition of energy-momentum tensor give the same expression for total energy-momentum, but it seems that energy-momentum is distributed between particle and the electromagnetic field in two different ways.

† For calculation of $\Theta_m{}^{\cdot i}{}_{\cdot k}$ one ought to pass to Euler's variables, to use (A.6) and to return to Lagrange's variables.

Let us take the standpoint (Klein, 1926, 1928; Jonsson, 1951; Rumer, 1956; Rylov, 1963), that real space-time is five-dimensional and is closed with respect to the fifth coordinate x^4. The x^4 is space-like, and corresponding to the x^4 canonical momentum of particles is electrical charge. Then expressions (A.7) and (A.10) are equivalent. The case is such that in a five-dimensional space-time the metric tensor γ^{AB} $(A, B = 0, 1, \ldots, 4)$ has the form

$$\begin{aligned} &\gamma^{ik} = g^{ik}, \qquad \gamma^{i4} = \gamma^{4i} = -g^{ik} A_k Q^{-1}, \\ &\gamma^{44} = -1 + A_i g^{ik} A_k Q^{-2} \qquad (i, k = 0, 1, 2, 3) \end{aligned} \tag{A.14}$$

where Q is some universal constant. In the five-space, canonical energy-momentum-charge tensor has the form $\{\Theta_{m\cdot k}^{\cdot i \cdot}, \Theta_{m\cdot 4}^{\cdot i \cdot}\}$, $\Theta_{m\cdot k}^{\cdot i \cdot}$ is obtained by means of (A.10), $\Theta_{m\cdot 4}^{\cdot i \cdot}$ describes four-current and has the form

$$\Theta_{m\cdot 4}^{\cdot i \cdot} = \frac{e}{c} \dot{q}^i \frac{\delta(q - x)}{|\dot{q}^0|} Q \tag{A.15}$$

Raising the last index in $\Theta_{m\cdot A}^{\cdot i \cdot}$ by means of γ^{AB} one gets

$$\Theta_m^{\cdot ik} = g^{kl} \Theta_{m\cdot l}^{\cdot i \cdot} + \gamma^{k4} \Theta_{m\cdot 4}^{\cdot i \cdot} = T_m^{\cdot ik} \tag{A.16}$$

Thus from the standpoint of five-dimensional space-time, (A.7) and (A.10) are two different forms of the same expression. Equation (A.7) is a gauge-invariant expression and has an advantage over (A.10).

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